PRECANONICAL QUANTIZATION AND THE SCHRÖDINGER WAVE FUNCTIONAL REVISITED

Igor V. Kanatchikov

National Quantum Information Centre in Gdańsk (KCIK)

ul. Wł. Andersa 27, 81-824 Sopot, Poland

Abstract

We address the long-standing issue of the relation between the Schrödinger functional representation in quantum field theory and the approach of precanonical field quantization which requires neither a distinguished time variable nor infinite-dimensional spaces of field configurations. The functional Schrödinger equation is derived in the limiting case $\varkappa \to \delta(0)$ from the Dirac-like covariant generalization of the Schrödinger equation within the precanonical quantization approach, where the constant $\varkappa$ of the dimension of the inverse spatial volume naturally appears on dimensional grounds. An explicit expression of the Schrödinger wave functional as a continuous product of precanonical wave functions on the finite-dimensional covariant configuration space of the field and space-time variables is obtained.

In loving memory of Pavel Efimov.

1 Introduction

Precanonical quantization of field theory was proposed in [1–3] as an analogue of canonical quantization which does not distinguish between space and time variables, and hence is more compliant with the relativity theory. It is inspired by the De Donder-Weyl (DW) analogue of the Hamiltonian formalism in the calculus of variations [5–8] which can be viewed as a generalization of the Hamiltonian description to field theory such that all space-time variables are treated on equal footing as analogues of the time parameter in mechanics, thus allowing us to view fields as finite-dimensional systems *changing* in space-time rather than infinite-dimensional mechanical systems *evolving* in time.

Specifically, given a first order Lagrangian function $L(y^a, y^a_\mu, x^\nu)$ on the space of field variables y^a, their first space-time derivatives y^a_μ, and space-time variables x^μ, one can introduce the Hamiltonian-like variables: $p^\mu_a := \partial L/\partial y^a_\mu$ (*polymomenta*) and $H = H(y^a, p^\mu_a, x^\nu) := y^a_\mu p^\mu_a - L$ (*DW Hamiltonian function*), and write the

Euler-Lagrange field equations in the Hamiltonian-like De Donder-Weyl form [5–8]

$$\partial_\mu y^a(x) = \frac{\partial H}{\partial p^\mu_a}\ , \quad \partial_\mu p^\mu_a(x) = -\frac{\partial H}{\partial y^a}\ , \tag{1.1}$$

provided $\det\left(\frac{\partial^2 L}{\partial y^a_\mu \partial y^b_\nu}\right) \neq 0$.

In this formulation the analogue of the (extended) configuration space is the finite-dimensional space of field variables y^a and space-time variables x^μ, and the analogue of the phase space is the polymomentum phase space of variables (y^a, p^μ_a, x^μ). Note that when the number of space-time dimensions $n = 1$ this formulation reduces to the familiar Hamiltonian formulation in mechanics. At the same time it provides an alternative to the standard extension of the Hamiltonian formulation to field theory which is based on the space-time decomposition and infinite-dimensional spaces of field configurations, and hence is limited to the globally hyperbolic space-times only.

The term "precanonical" stems from the observation that the symplectic structure (and other structures) of the canonical formalism can be related to, or derived from, those of the De Donder-Weyl (or polysymplectic, or multisymplectic) formalism by restricting the latter to the surface of initial field configurations at a fixed moment of time (which we call the Cauchy surface) and integrating over it [9–15].

Though the idea of an approach to field quantization based on the Hamiltonian-like formulation above was discussed already by Hermann Weyl himself in 1934 [16], it was not further developed because an analogue of the Poisson bracket in the DW Hamiltonian formalism was not known until it was constructed using the polysymplectic structure on the polymomentum phase space in [9, 17, 18] (see also [19–25] for later discussions and generalizations).

Since the Poisson bracket in precanonical DW formulation [9, 17, 18] is defined on differential forms representing the dynamical variables in field theory, and leads to the Gerstenhaber algebra structure (graded Lie bracket + Grassmann product with the respective grades differing by 1) generalizing the standard Poisson algebra structure, its quantization has been shown to lead to a quantum formalism with Clifford-valued operators and wave functions, and the following precanonical generalization of the Schrödinger equation [1–3]

$$i\hbar\varkappa\gamma^\mu\partial_\mu\Psi = \widehat{H}\Psi, \tag{1.2}$$

where $\Psi = \Psi(y^a, x^\mu)$ is the Clifford-valued wave function on the covariant configuration space, $\widehat{H}$ is the operator of the DW Hamiltonian function, which involves partial derivatives ∂_{y^a} with respect to the field variables, and $\varkappa$ is a (large) constant of the dimension $\mathtt{L}^{-(n-1)}$ in n space-time dimensions which appears on dimensional grounds when, e.g. the differential form corresponding to the infinitesimal volume element of space $d\mathbf{x} = dx^1 \wedge ... \wedge dx^{n-1}$ is mapped to the corresponding element of the space-time Clifford algebra, viz.

$$d\mathbf{x} \xrightarrow{q} \frac{1}{\varkappa}\gamma_0, \tag{1.3}$$

using the "quantization map" q known in the theory of Clifford algebras (see e.g. [26]).

In a sense, as a consequence of precanonical quantization, $\varkappa$ effectively introduces in the theory a "minimal volume" scale $1/\varkappa$ without any a priori assumptions regarding the microscopic structure of space-time. The limit of the vanishing "minimal volume" scale corresponds to $1/\varkappa \to 0$ or, more precisely,

$$\frac{1}{\varkappa}\gamma_0 \xrightarrow{q^{-1}} d\mathbf{x}$$

or, equivalently,

$$\gamma^0 \varkappa \xrightarrow{q^{-1}} \delta^{n-1}(\mathbf{0}), \tag{1.4}$$

where q^{-1} is the inverse of the "quantization map".

The precanonical wave function $\Psi(y,x)$ can be interpreted as the probability amplitude of finding the value of the field in the interval $[y, y+dy]$ when observed in the vicinity of the space-time point $[x, x+dx]$. This interpretation is suggested by the conservation law

$$\partial_\mu \int dy \, \mathsf{Tr}(\overline{\Psi}\gamma^\mu \Psi) = 0,$$

which follows from (1.2) if $\widehat{H}$ is self-adjoint with respect to the scalar product $\int dy \mathsf{Tr}(\overline{\Phi}\Psi)$, and the positive definiteness of the Frobenius norm $\mathsf{Tr}(\Psi^\dagger\Psi)$.

Note that at $n = 1$ the formulation of precanonical quantization reduces to the conventional quantum mechanics.

The description of quantum fields based on precanonical quantization appears to be fundamentally different from the familiar formulations of quantum field theory. By avoiding the explicit treatment of fields as infinite-dimensional Hamiltonian systems and replacing the starting point of quantization with the covariant De Donder-Weyl Hamiltonian formalism an obvious connection is lost with the concepts of the standard formulations of QFT, such as free particles, which straightforwardly follow from the conventional treatment and are crucial for the comparison of the results of quantum field theory with the experiments, at least in the perturbative regime. On the other hand, the construction of precanonical quantization is non-perturbative, explicitly compliant with the relativistic principles, and it seems to be potentially better or more easily defined mathematically than the infinite-dimensional constructions in QFT.

In spite of the fact that there already have been attempts to apply precanonical quantization in quantum Yang-Mills theory [27], quantum gravity [28–33] and string theory [34], the lack of good understanding of the connections of the description suggested by precanonical quantization with the concepts of standard QFT has hindered its applications and further development so far.

Note also that several other attempts of field quantization inspired by the covariant Hamiltonian-like formulations in the calculus of variations can be found in the recent literature, see e.g. [35–40].

In our previous papers [14, 27] we tried to understand the connection of precanonical quantization and its description of quantum fields in terms of Clifford-valued wave *functions* $\Psi(y, x)$ with the standard canonical quantization in the functional Schrödinger representation, where the states of quantum fields are described by the wave *functional* $\mathbf{\Psi}([y(\mathbf{x})], t)$ on the infinite-dimensional configuration space of field configurations $y(\mathbf{x})$ at the instant of time t. However, the discussion in [1, 14, 27] does not establish a convincing connection because of the following shortcomings of the argument:

(i) the simplifying ultra-locality assumption that the Schrödinger functional $\mathbf{\Psi}([y(\mathbf{x})], t)$ can be represented in the form of the continuous product of precanonical wave functions $\Psi(y, \mathbf{x}, t)$ over all points of the surface $y = y(\mathbf{x})$ (cf. eq. (2.2) below) neglects the correlations between the field values in the space-like separated points, thus contradicting both the known explicit form of the exact solutions of the functional Schrödinger equation for free field theories [41, 42, 44] and the known behaviour of the Wightman functions of free fields;

(ii) an attempt to take those correlations into account by means of a functional "unitary transformation" which would transform the equation in functional derivatives satisfied by the ultralocal continuous product Ansatz of [14] into the equation for the wave functional in the Schrödinger picture leads to a non-integrable equation in functional derivatives $\delta \mathbf{N}/\delta y(\mathbf{x}) = \gamma^i \partial_i y(\mathbf{x})$ for the functional operator $e^{i\mathbf{N}}$ determining such a transformation;

(iii) the representation of the continuous product $\prod_{\mathbf{x}} \Psi(\mathbf{x})$ as $e^{\left(\varkappa \int d\mathbf{x} \ln(\Psi(\mathbf{x}))\right)}$ used in [14] is questionable for Clifford-valued $\Psi(\mathbf{x})$ which may not commute at different points $\mathbf{x}$; besides, its functional differentiation implies the existence of the inverse $\Psi^{-1}(\mathbf{x})$ for all $\mathbf{x}$, which is too restrictive and even impossible to define if Ψ is spinor-valued.

In this paper we revisit the earlier treatment [14] of the relation between the precanonical wave function and the Schrödinger wave functional using the example of scalar field theory. We follow the conventions and notations of that paper, as well as we refer to it both for brief outline of the elements of precanonical and canonical quantization of scalar field theory and explanation of some constructions to be used here. It will be shown below that a minimal modification of the ultra-local Ansatz used in [1, 14, 27] allows us to take into account the space-like correlations, which were neglected in the treatment of [14], and to derive a formula expressing the Schrödinger wave functional in terms of precanonical wave functions.

2 Precanonical wave functions and the Schrödinger wave functional

We restrict ourselves to the example of the scalar field theory given by (henceforth we assume ($\hbar = 1$)):

$$L = \frac{1}{2}\partial_\mu y \partial^\mu y - V(y),$$

where the potential term also includes the mass term $\frac{1}{2}m^2y^2$. The Schrödinger wave functional for the quantum scalar field obeys the Schrödinger equation in functional derivatives [41–45]

$$i\partial_t \mathbf{\Psi} = \int d\mathbf{x} \left\{ -\frac{1}{2}\frac{\delta^2}{\delta y(\mathbf{x})^2} + \frac{1}{2}(\nabla y(\mathbf{x}))^2 + V(y(\mathbf{x})) \right\} \mathbf{\Psi}. \tag{2.1}$$

Our problem is to clarify how the description of quantum fields in terms of the wave functional $\mathbf{\Psi}([y(\mathbf{x})], t)$ is related to the description in terms of the precanonical wave function $\Psi(y, x)$, and how the Schrödinger equation of the canonical quantization approach, eq. (2.1), is related to the Dirac-like partial differential equation, eq. (1.2), playing the role of the Schrödinger equation for the precanonical wave function.

If the wave function $\Psi(y, x)$ of the precanonical quantization approach has the probabilistic interpretation as the probability amplitude of finding the value y of the field at the space-time point x, then the Schrödinger wave functional $\mathbf{\Psi}([y(\mathbf{x})], t)$, which is the probability amplitude of observing the field configuration $y(\mathbf{x})$ on the space-like hypersurface of constant time t, should be given by a certain composition of amplitudes $\Psi(y, \mathbf{x}, t)$ taken along the Cauchy surface $\Sigma : (t = \text{const}, \ y = y(\mathbf{x}))$ in the covariant configuration space of variables (y, x).

If we assume that the probability amplitudes of observing the field values y are independent in space-like separated points, then $\mathbf{\Psi}([y(\mathbf{x})], t)$ is given by the product over all points of Σ of the wave function $\Psi(y, x)$ restricted to Σ: $\Psi(y, x)|_\Sigma = \Psi_\Sigma(y = y(\mathbf{x}), \mathbf{x}, t)$, i.e.

$$\mathbf{\Psi} \sim \prod_{\mathbf{x}\in\Sigma} \Psi_\Sigma(y = y(\mathbf{x}), \mathbf{x}, t). \tag{2.2}$$

This ultra-locality assumption is, however, unphysical and an improved representation of the Schrödinger wave functional in terms of $\Psi(y, x)|_\Sigma$ has to be found, which would take into account the correlations of the amplitudes $\Psi(y, x)$ in space-like separated points.

The task is similar to the probability theory where the joint probability of two events A and B is given in general by $P(A, B) = P(A|B)P(B)$, where $P(A|B)$ is the conditional probability of A given B, which reduces to $P(A)$ only if the events A and B are independent. In our case we have a continuum of events of obtaining the values $y_\mathbf{x}$ of the field in the corresponding points $\mathbf{x}$ of the hypersurface of constant time t and their respective probability amplitudes $\Psi(y_\mathbf{x}, \mathbf{x}, t)$ given by $\Psi(y = y(\mathbf{x}), \mathbf{x}, t)$. As a minimal deviation from the simplest ultralocal product formula in (2.2) let us assume that the correlations between space-like separated points can be taken into account by a multiplication of $\Psi(y, x)|_\Sigma$ by some function of the field configuration denoted $U(y(\mathbf{x}))$, so that the Schrödinger functional can be given by a modified product formula $\mathbf{\Psi} \sim \prod_{\mathbf{x}\in\Sigma} U(y(\mathbf{x}))\Psi(y, x)|_\Sigma$. Below we show that this minimal modification of the ultra-local product formula of [14] is general enough to formulate a connection between the precanonical description of quantum fields and the canonical description in the functional Schrödinger representation.

Thus, let us assume that the Schrödinger wave functional has the form

$$\mathbf{\Psi} = \mathsf{Tr}\left\{\prod_{\mathbf{x}} U(y(\mathbf{x}))\Psi_\Sigma(y(\mathbf{x}), \mathbf{x}, t)\right\}, \tag{2.3}$$

where $U(y(\mathbf{x}))$ is a matrix transformation which depends on the value $y(\mathbf{x})$ and its derivatives in the point $\mathbf{x}$ (in order to go beyond ultra-locality). The latter expression means that for any $\mathbf{x}$ the functional $\mathbf{\Psi}$ can be written as

$$\mathbf{\Psi} = \mathsf{Tr}\left\{\Xi(\breve{\mathbf{x}}, t)U(y(\mathbf{x}))\Psi_\Sigma(y(\mathbf{x}), \mathbf{x}, t)\right\}, \tag{2.4}$$

where

$$\Xi(\breve{\mathbf{x}}, t) := \prod_{\mathbf{x}'\neq\mathbf{x}} U(y(\mathbf{x}'))\Psi_\Sigma(y(\mathbf{x}'), \mathbf{x}', t) \tag{2.5}$$

and the continuous product here implies a symmetrization over all points $\mathbf{x}'$.

This observation facilitates the calculation of functional derivatives of $\mathbf{\Psi}$, viz.

$$\frac{\delta\mathbf{\Psi}}{\delta y(\mathbf{x})} = \mathsf{Tr}\left\{\Xi(\breve{\mathbf{x}})\frac{\delta U(\mathbf{x})}{\delta y(\mathbf{x})}\Psi_\Sigma(\mathbf{x}) + \Xi(\breve{\mathbf{x}})U(\mathbf{x})\delta(\mathbf{0})\partial_y\Psi_\Sigma(\mathbf{x})\right\}, \tag{2.6}$$

and

$$\begin{aligned}\frac{\delta^2\mathbf{\Psi}}{\delta y(\mathbf{x})^2} = \mathsf{Tr}\Bigg\{&\Xi(\breve{\mathbf{x}})\frac{\delta^2 U(\mathbf{x})}{\delta y(\mathbf{x})^2}\Psi_\Sigma(\mathbf{x}) + \Xi(\breve{\mathbf{x}})U(\mathbf{x})\delta(\mathbf{0})^2\partial_{yy}\Psi_\Sigma(\mathbf{x})\\ &+2\,\Xi(\breve{\mathbf{x}})\frac{\delta U(\mathbf{x})}{\delta y(\mathbf{x})}\delta(\mathbf{0})\partial_y\Psi_\Sigma(\mathbf{x})\Bigg\},\end{aligned} \tag{2.7}$$

where the shorthand notations $\Xi(\breve{\mathbf{x}})$ for $\Xi(\breve{\mathbf{x}}, t)$, $U(\mathbf{x})$ for $U(y(\mathbf{x}))$, and $\Psi_\Sigma(\mathbf{x})$ for $\Psi_\Sigma(y(\mathbf{x}), \mathbf{x}, t)$ are introduced. Note that the $(n-1)$-dimensional $\delta(\mathbf{0})$ appears here as a result of functional differentiation of a function with respect to itself at the same point.

For the derivative of $\mathbf{\Psi}$ with respect to the time variable we use the chain rule

$$\partial_t\mathbf{\Psi} = \mathsf{Tr}\left\{\int d\mathbf{x}\ \frac{\delta\mathbf{\Psi}}{\delta\Psi^T_\Sigma(\mathbf{x})}\partial_t\Psi_\Sigma(\mathbf{x})\right\}, \tag{2.8}$$

where Ψ^T denotes the transpose of Ψ. Using (2.4) we obtain:

$$i\partial_t\mathbf{\Psi} = \mathsf{Tr}\left\{\int d\mathbf{x}\ \Xi(\breve{\mathbf{x}})U(\mathbf{x})\delta(\mathbf{0})i\partial_t\Psi_\Sigma(\mathbf{x})\right\}. \tag{2.9}$$

Thus the time evolution of the wave functional $\mathbf{\Psi}$ is totally dictated by the time evolution of the precanonical wave function restricted to the Cauchy surface, $\Psi_\Sigma(\mathbf{x})$.

The latter is given by our Dirac-like precanonical Schrödinger equation on $\Psi(y, x)$, eq. (1.2), restricted to the Cauchy surface Σ, so that (cf. [14]):

$$i\partial_t\Psi_\Sigma(\mathbf{x}) = -i\beta\gamma^i\frac{d}{dx^i}\Psi_\Sigma(\mathbf{x}) + i\beta\gamma^i\partial_i y(\mathbf{x})\partial_y\Psi_\Sigma(\mathbf{x}) + \frac{1}{\varkappa}\beta(\widehat{H}\Psi)_\Sigma(\mathbf{x}). \tag{2.10}$$

Here $\beta := \gamma^0$, $\frac{d}{dx^i}$ denotes the total derivative along Σ:

$$\frac{d}{dx^i} := \partial_i + \partial_i y(\mathbf{x})\partial_y + \partial_{ij} y(\mathbf{x})\partial_{y_j} + ..., \tag{2.11}$$

and, in the specific case of the scalar field y,

$$(\widehat{H}\Psi)_\Sigma(\mathbf{x}) = -\frac{1}{2}\varkappa^2 \partial_{yy}\Psi_\Sigma(\mathbf{x}) + V(y(\mathbf{x}))\Psi_\Sigma(\mathbf{x}) \tag{2.12}$$

(see [1–3, 14] for more details).

It is easy to see that if equation (2.10) is substituted in (2.9) the potential term in (2.12) yields

$$\mathsf{Tr}\left\{ \int d\mathbf{x}\ \Xi(\check{\mathbf{x}})U(\mathbf{x})\delta(\mathbf{0})\frac{1}{\varkappa}\beta V(y(\mathbf{x}))\Psi_\Sigma(\mathbf{x}) \right\}.$$

It will reduce to the potential term in the functional Schrödinger equation:

$$\int d\mathbf{x} V(y(\mathbf{x}))\mathbf{\Psi},$$

with $\mathbf{\Psi}$ given by (2.4), if $\varkappa\beta$ goes over into $\delta(\mathbf{0})$:

$$\varkappa\beta \to \delta(\mathbf{0}). \tag{2.13}$$

Under the same condition the term $\varkappa^2\partial_{yy}\Psi_\Sigma$ in (2.12) reproduces the second term in the second functional derivative of $\mathbf{\Psi}$, eq. (2.7).

Thus, the relation (2.13) establishes a condition under which the transition from precanonical to the functional Schrödinger description is possible. We note that it coincides with the inverse quantization map and the vanishing minimal volume limit in eq. (1.4).

Next, we note that in order to obtain a description in terms of the wave functional $\mathbf{\Psi}$ the remaining terms in front of $\partial_y\Psi_\Sigma$ in (2.10) and (2.7) should cancel each other. It leads to the equation

$$U(\mathbf{x})i\beta\gamma^i\partial_i y(\mathbf{x}) + \frac{\delta U(\mathbf{x})}{\delta y(\mathbf{x})} = 0 \tag{2.14}$$

which determines $U(\mathbf{x})$. By taking into account the condition (2.13) again, in the corresponding limit we can write (2.14) as

$$U(\mathbf{x})i\gamma^i\partial_i y(\mathbf{x})\delta(\mathbf{0}) + \varkappa\frac{\delta U(\mathbf{x})}{\delta y(\mathbf{x})} = 0. \tag{2.15}$$

The solution of this equation can be written in the form

$$U(\mathbf{x}) = e^{-iy(\mathbf{x})\gamma^i\partial_i y(\mathbf{x})/\varkappa} \tag{2.16}$$

(up to an integrating factor which is implicitly taken into account in $\Xi(\breve{\mathbf{x}})$). Besides, from (2.15), in the limit (2.13), it also follows that

$$\frac{\delta^2 U(\mathbf{x})}{\delta y(\mathbf{x})^2} = (\nabla y(\mathbf{x}))^2 U(\mathbf{x}). \tag{2.17}$$

Hence, the first term in (2.7) correctly reproduces the term $\int d\mathbf{x}(\nabla y(\mathbf{x}))^2 \mathbf{\Psi}$ in the functional derivative Schrödinger equation, eq. (2.1).

The last term to be considered is the total derivative term in (2.10). When inserted in (2.8), with the condition (2.13) taken into account, and then integrated by parts, it yields a term proportional to

$$\mathsf{Tr}\left\{\int d\mathbf{x}\ \Xi(\breve{\mathbf{x}})\frac{d}{dx^i}U(\mathbf{x})\gamma^i\Psi_\Sigma(\mathbf{x})\right\}.$$

Using the explicit form of $U(\mathbf{x})$ in (2.16) to evaluate $\frac{d}{dx^i}U(\mathbf{x})$ and recalling the representation of $\mathbf{\Psi}$ in (2.4), this term is transformed to the form

$$\int d\mathbf{x}\ [y(\mathbf{x})\gamma^i\gamma^j\partial_{ij}y(\mathbf{x}) + \gamma^i\partial_i y(\mathbf{x})\gamma^j\partial_j y(\mathbf{x})]\mathbf{\Psi},$$

which obviously vanishes upon integrating by parts. Consequently, the total derivative term in (2.10) does not contribute to the functional derivative Schrödinger equation on $\mathbf{\Psi}$.

Thus, we have demonstrated that the substitution of the precanonical Schrödinger equation restricted to the Cauchy surface Σ, eqs. (2.10–2.12), into the expression (2.8) for the time derivative of the functional (2.3) constructed from precanonical wave functions allows us to obtain all terms which are present in the functional Schrödinger equation and cancels those missing there. Besides, it fixes both the condition (2.13) under which the transition from precanonical to the canonical functional description is possible and the transformation U in (2.3) which then correctly reproduces the $\frac{1}{2}(\nabla y(\mathbf{x}))^2$ term in the canonical Hamiltonian, which is the term responsible for the non-ultralocal behaviour of quantum fields (cf. e.g. [46]). The procedure also yields an explicit limiting expression of the Schrödinger wave functional as the continuous product over all points of space of U-transformed precanonical wave functions restricted to the Cauchy surface Σ, viz.

$$\mathbf{\Psi}([y(\mathbf{x})],t) = \mathsf{Tr}\left\{\prod_{\mathbf{x}} e^{-iy(\mathbf{x})\gamma^i\partial_i y(\mathbf{x})/\varkappa}\Psi_\Sigma(y(\mathbf{x}),\mathbf{x},t)\right\}, \tag{2.18}$$

which is valid in the limit of infinitesimal "minimal volume" $1/\varkappa$, when $\varkappa\beta \xrightarrow{q^{-1}} \delta(\mathbf{0})$.

3 Conclusions

We have shown how the functional derivative Schrödinger equation (2.1) can be derived from the partial differential covariant precanonical Schrödinger equation

restricted to the Cauchy surface in the covariant configuration space of field theory. We also obtained an explicit limiting expression of the Schrödinger wave functional of the canonical quantization approach in terms of the precanonical wave functions defined on the finite dimensional space of field and space-time variables. The result suggests that quantum field theory resulting from canonical quantization can be viewed as a limit $1/\varkappa \to 0$ of quantum field theory resulting from the precanonical quantization approach.

Note that the result of our earlier discussion in [14] comes amazingly close to the formula (2.18).

First, we found that the transformation which allows us to fully take into account the deviations from ultra-locality acts directly on precanonical wave functions rather than on the ultra-local functional constructed from them. This solves the problem with the non-integrability of the functional derivative equation for the corresponding transformation functional, which we tried to circumvent in [14, 27].

Second, we found that the Schrödinger functional description is possible under the limiting condition (2.13), which essentially tells us that the description of quantum fields in terms of the Schrödinger wave functional corresponds to the limit of vanishing "minimal volume" $1/\varkappa$: $\varkappa \to \delta(\mathbf{0})$, as it was already noticed in [14]. It should be noted that the condition (2.13) complies both with the relativistic transformation laws of $\delta(\mathbf{0})$ and the precanonical quantization itself (cf. eq.(1.3)). Namely, (2.13) unifies two requirements to be fulfilled when a transition from precanonical to the functional Schrödinger description is being made: the absolute value of $\varkappa$ should tend to $\delta(\mathbf{0})$ and the inverse of the "quantization map" in (1.3) should be applied so that the Clifford valued operator of the "minimal volume" $\beta/\varkappa$ goes over (at $\varkappa \to \delta(\mathbf{0})$) into the classical infinitesimal volume element $d\mathbf{x}$.

Third, the identification of the condition when the transformation from precanonical to the functional Schrödinger representation is possible as the inverse quantization map in the limit $1/\varkappa \to 0$, eq. (1.4), has made superficial the use of the projector $\frac{1}{2}(1+\beta)$ introduced in the expression of the wave functional in [14].

In mathematical terms, by starting from the assumption (2.3) and showing how it allows us to derive the canonical functional derivative Schrödinger equation from the precanonical partial derivative Schrödinger equation, and to fix the form of the transformation $U(y(\mathbf{x}))$, we have proven that the Ansatz (2.3) is the sufficient condition. In a separate paper we will show that it is also necessary.

Note that although our result is obtained using the particular case of the scalar field theory, in the subsequent papers it will be demonstrated that it can be extended also to other fields, such as Yang-Mills and spinor fields.

One should underline that the nature of the parameter $\varkappa$ appearing in precanonical quantization is different from the "minimal length" scale introduced by hand in nonlocal or discrete/microstructured space-time field theories. For example, it was found to disappear from the final results for free field theories. We still have to investigate in detail what the role of $\varkappa$ is in interacting field theories, renormalizable and non-renormalizable, and what is its possible role in the common renormalization

techniques.

In conclusion, let us recall that the main motivation of our developing of the precanonical quantization approach has been its potential to be a better synthesis of relativity and quantum theory in field theory, which could provide a better framework for quantization of gravity. We hope to use the results of this paper for further development of precanonical quantization of gravity [28–31] and, in particular, for better understanding of its relation with the canonical quantization leading to the Wheeler-DeWitt equation, the quantum gravity analogue of the functional Schrödinger equation.

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